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# Note

## Maximum Non-path-connected Graphs

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Let $n$ and $i$ be integers with $n \geqslant 4$ and $2 \leqslant i \leqslant n-1$. Lewin (*J. Combin. Theory Ser. B* **25** (1978), 245–257) has determined the smallest size of a connected $n$-graph without end vertices which ensures the existence of a path of length $i$ between every pair of distinct vertices of the graph. Here all the connected $n$-graphs without end vertices of maximum size are found which fail to have a path of length $i$ between every pair of distinct vertices. © 1984 Academic Press, Inc.

### 1. Introduction and Notation

Graph-theoretic terminology follows that of references [1] and [4]. In particular: $V(G)$ and $E(G)$ denote, respectively, the vertex set and edge set of the graph $G$; $|V(G)|$ is the *order* of $G$ and $|E(G)|$ is the *size* of $G$; an *n-graph* is a graph of order $n$ and an $(n, q)$*-graph* is an $n$-graph of size $q$; if $v \in V(G)$ then denote by $N(v)$ the set of vertices adjacent to $v$; $|N(v)|$ is the *degree* of $v$, denoted $\deg v$; the *minimum vertex degree* in $G$ is denoted $\delta(G)$; if $v \in V(G)$ then $G-v$ is the graph obtained from $G$ by deleting $v$ and its incident edges. We also use the following notation: for $m$, $n \geqslant 2$, $K_m \cdot K_n$ is the graph of order $m+n-1$ consisting of a copy of $K_m$ and a copy of $K_n$ with exactly one vertex in common; for $m$, $n \geqslant 3$, $K_m|K_n$ is the graph of order $m+n-2$ consisting of a copy of $K_m$ and a copy of $K_n$ with exactly two vertices in common; the greatest integer function is denoted $\lfloor\ \rfloor$; following [5] we define $n_j = \binom{n-j}{2}$ for integer $j \geqslant 0$.

A graph $G$ is called *hamiltonian-connected* if there is a hamiltonian path between every pair of distinct vertices of $G$. This concept was generalized in [3] to that of path-connectedness: if $G$ is an $n$-graph and $i$ is an integer with $2 \leqslant i \leqslant n-1$, then $G$ is $P_i$*-connected* (or $P_i$ *holds in* $G$) if there is a path of length $i$ between every pair of distinct vertices of $G$. If there is a $u-v$ path of length $i$ in $G$, then we will say that $P_i(u, v)$ holds in $G$.

The first theorem is due to Ore [6, Theorems 4.1 and 5.1].



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THEOREM A. *Let $G$ be an $(n, q)$-graph with $n \geqslant 4$ and $q \geqslant n_1 + 2$. If $G$ is not hamiltonian-connected then $q = n_1 + 2$ and $G = K_3 | K_{n-1}$ or $K_3 + \bar{K}_3$.*

It is natural to ask for analogous results relating to path-connectedness. However, a graph of order $n \geqslant 4$ which is either disconnected or has an end vertex cannot be $P_i$-connected for any $i$. Therefore we will restrict our attention to connected graphs without end vertices and will call such graphs *admissible*.

Let $n$ and $i$ be integers with $n \geqslant 4$ and $2 \leqslant i \leqslant n-1$. Let $m(n, i)$ denote the smallest integer such that if $G$ is an admissible $(n, q)$-graph with $q \geqslant m(n, i)$ then $G$ is $P_i$-connected. Let $l(n, i) = m(n, i) - 1$ and let $L(n, i)$ denote the (nonempty) set of admissible $(n, l(n, i))$-graphs which are not $P_i$-connected. We can now regard Theorem A as determining $m(n, n-1)$ and $L(n, n-1)$.

Lewin has found all other values of $m(n, i)$ and we refer the reader to Table I of [5] for these values. He also supplies at least one member of $L(n, i)$ for each $n$ and $i$ within the relevant boundaries. Our object here is to find all the graphs in $L(n, i)$ for $n \geqslant 4$, $2 \leqslant i \leqslant n-2$. Other extremal problems on path-connectedness have been considered by Enomoto and Usami [2, 7, 8].

## 2. RESULTS

THEOREM 1. *For $n \geqslant 4$,*

$$L(n, 2) = \{(K_1 \cup K_k) + (K_1 \cup K_{n-k-2}) \colon 1 \leqslant k \leqslant \lfloor (n-2)/2 \rfloor\}.$$

*Proof.* Suppose that $G \in L(n, 2)$ and that $P_2(u, v)$ does not hold in $G$ where, without loss of generality, $\deg u \geqslant \deg v = k + 1 \geqslant 2$. Thus $N(u) \cap N(v) = \varnothing$, $\deg u + \deg v \leqslant n$ and $k \leqslant \lfloor (n-2)/2 \rfloor$. We now have

$$n_1 + 1 = l(n, 2) = |E(G)| \leqslant (n + (n-2)(n-2))/2 = n_1 + 1. \tag{1}$$

Equality must hold throughout (1) so that $\deg u + \deg v = n$, $uv \in E(G)$, $G - u - v = K_{n-2}$, and every vertex in $G - u - v$ is adjacent to exactly one of $u$ and $v$. Hence $G = (K_1 \cup K_k) + (K_1 \cup K_{n-k-2})$ and the result follows since this graph belongs to $L(n, 2)$ [5, p. 247]. ∎

THEOREM 2. *For $n \geqslant 5$,*

$$L(n, 3) = \{\bar{K}_2 + (K_2 \cup K_{n-4}), \bar{K}_3 + (K_2 \cup K_{n-5})\},$$

*where $K_2 \cup K_0 = K_2$.*

*Proof.* It is easily checked that these graphs (the first of which was given by Lewin) belong to $L(n, 3)$. Conversely suppose $G \in L(n, 3)$ and that

$P_3(u, v)$ does not hold in $G$. Let $N'_u$, $N'_v$, $N'_{uv}$ denote the sets of vertices in $V(G) - \{u, v\}$ which are adjacent to $u$ but not $v$, $v$ but not $u$, both $u$ and $v$, respectively. Let $|N'_u| = r$, $|N'_v| = s$, and $|N'_{uv}| = t$. Since $\delta(G) \geqslant 2$ we have $r + t \geqslant 1$ and $s + t \geqslant 1$. Since $P_3(u, v)$ does not hold in $G$, there can be no edge in $G - u - v$ connecting a neighbor of $u$ to a neighbor of $v$. Therefore

$$\begin{aligned} n_2 + 4 = l(n, 3) &= |E(G)| \\ &\leqslant 1 + r + s + 2t + n_2 - rs - rt - st - t_0. \end{aligned} \tag{2}$$

The expression on the right-hand side of (2) has a maximum value of $n_2 + 4$ which is attained if and only if $r = s = 0$ and $t = 2$ or 3. It follows that $G$ is one of the required graphs. ∎

THEOREM 3. (a) $L(7, 5) = \{K_4 | K_5, K_3 + \bar{K}_4\}$.

(b) $L(9, 7) = \{K_4 | K_7, K_4 + \bar{K}_5\}$.

(c) *For* $n \geqslant 6$, $n \neq 7$ *or* 9, $L(n, n-2) = \{K_4 | K_{n-2}\}$.

THEOREM 4. (a) $L(7, 4) = \{K_3 \cdot K_5, K_2 + (2K_2 \cup K_1)\}$.

(b) $L(8, 5) = \{K_5 | K_5\}$.

(c) $L(9, 6) = \{K_3 \cdot K_7, K_5 | K_6\}$.

(d) *For* $n \geqslant 8$, $4 \leqslant i \leqslant n-3$, $(n, i) \neq (8, 5)$ *or* $(9, 6)$, *we have* $L(n, i) = \{K_3 \cdot K_{n-2}\}$.

The proofs of these results run along similar lines and so, for the sake of brevity, we will only give the details for Theorem 4. To facilitate this, we make a simple observation.

LEMMA 1. *If* $P_{i-1}$ *and* $P_i$ *hold in* $G - v$ *and* $\deg_G v \geqslant 2$ *then* $P_i$ *holds in* $G$.

*Proof of Theorem* 4. The sets $L(n, n-3)$, $7 \leqslant n \leqslant 12$, can be determined by ad hoc arguments and so we eliminate them from future consideration. In each remaining case $l(n, i) = n_2 + 3$ and $K_3 \cdot K_{n-2} \in L(n, i)$ [5, p. 254].

Conversely, suppose $G \in L(n, i)$ and that $P_i(u, v)$ does not hold in $G$, with $\deg u \geqslant \deg v$. Also suppose that $L(m, j)$ is known for all values of $m$ and $j$ such that $m \leqslant n$, $j \leqslant i$, and $m + j < n + i$.

If $G$ contains two vertices, $y$ and $z$ say, of degree two then $G - y - z = K_{n-2}$ or $K_{n-2} - e$ and it is easy to see that $P_i$ holds in $G$ unless $G = K_3 \cdot K_{n-2}$. Henceforth we assume that $G$ has at most one vertex of degree two. We will now derive a contradiction, from which the desired result follows. Let $x$ be a vertex of minimum degree in $G$.

*Case* 1. Suppose $i = 4$. If $\delta(G) \leqslant n - 4$ then $G - x$ is an admissible

$(n-1)$-graph with at least $(n-1)_2+4$ edges. Therefore $P_4$ holds in $G-x$. By Lemma 1, $P_3$ does not hold in $G-x$ and therefore $|E(G-x)| = (n-1)_2+4$ and $\delta(G) = \deg x = n-4$. If every vertex of $V(G)-\{u,v\}$ has degree $\geqslant n-3$ in $G$ then, since $n \geqslant 8$, we have

$$|E(G)| \geqslant (2(n-4)+(n-2)(n-3))/2 > n_2+3 = |E(G)|, \tag{3}$$

which is a contradiction. We may therefore assume that $x \notin \{u,v\}$. But now, since $P_4$ holds in $G-x$, $P_4(u,v)$ holds in $G$. This contradiction implies that $\delta(G) \geqslant n-3$, which produces a contradiction as in (3).

*Case* 2. Suppose $5 \leqslant i \leqslant n-4$. If $\delta(G) \leqslant n-4$ then $G-x$ is an admissible $(n-1)$-graph with at least $(n-1)_2+4$ edges. Therefore $P_{i-1}$ and $P_i$ hold in $G-x$ and, by Lemma 1, $P_i$ holds in $G$ with one possible exception when $(n,i)=(9,5)$, $G-x=K_5|K_5$, and $\delta(G)=\deg x = n-4=5$. However, it is easy to see that $G$ is also $P_5$-connected in this case, a contradiction. Therefore $\delta(G) \geqslant n-3$, which leads to a contradiction as in Case 1.

*Case* 3. Suppose $i=n-3$ with $n \geqslant 13$. If $\delta(G) \leqslant n-6$ then $G-x$ is an admissible $(n-1)$-graph with at least $(n-1)_2+6$ edges, in which case $P_{n-3}$ and $P_{n-4}$ hold in $G-x$ and, by Lemma 1, $P_{n-3}$ holds in $G$, a contradiction. If $\delta(G) \geqslant n-4$ then, since $n \geqslant 13$, we have $|E(G)| \geqslant n(n-4)/2 > n_2+3 = |E(G)|$, a contradiction. Therefore $\delta(G)=n-5$ and $G-x$ is an admissible $(n-1, (n-1)_2+5)$-graph. Therefore $P_{n-4}$ holds in $G-x$. By Lemma 1, $P_{n-3}$ does not hold in $G-x$ and so, by Theorem 3(c), $G-x=K_4|K_{n-3}$. But now $8 \leqslant n-5 = \delta(G) \leqslant 1+\delta(G-x)=4$, a final contradiction. ∎

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